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The effect of heating rate on the stability of stationary fluids

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A horizontal fluid layer whose lower surface temperature is made to vary with time is considered. The stability analysis for this situation shows that the criterion for the onset of instability in a fluid layer which is being heated from below, depends on both the method and the rate of heating. For a fluid layer with two rigid boundaries, the minimum Rayleigh number corresponding to the onset of instability is found to be 1340. For slower heating rates the critical Rayleigh number increases to a maximum value of 1707·8, while for faster heating rates the critical Rayleigh number increases without limit.

Two specific types of heating are investigated in detail, constant flux heating and linearly varying surface temperature. These cases correspond closely to situations for which published data exist. The results are in good qualitative agreement.

1. Introduction

It is well known that if a layer of stationary fluid is heated from below or cooled from above, a point is reached when the adverse density stratification causes an instability in the fluid. By solving the linearized equations perturbed about a stationary fluid with a uniform temperature gradient, considerable success has been realized in obtaining an analytical understanding of the basic phenomena associated with the instability. These results are summarized by Chandrasekhar (1961). If heat is introduced slowly, the instability usually manifests itself in the form of two-dimensional rolls, the value of the critical Rayleigh number being dependent on the form of the fluid boundaries.

On several occasions, however, it has been observed that thermally induced instabilities occur under conditions which are at variance with the existing theories. It was observed by Graham (1933) that a columnar mode of instability could be established at a smaller Rayleigh number than that predicted by the theory, but no numerical data were reported. Measurements of the minimum temperature differences required for maintained columnar convection were recorded by Chandra (1938) and by Sutton (1950). These temperature differences are considerably smaller than the values predicted theoretically, particularly at

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small fluid depths. It was suggested by Sutton that the non-uniform temperature gradient, caused by rapid heating, might be the cause of columnar instability.

While measuring the rate of heat transfer across layers with two rigid boundaries, de Graaf & van der Held (1953) noted that columnar instability could be initiated at Rayleigh numbers as low as 1400, rather than 1707·8 as predicted by the existing theory. These authors also agreed with Sutton that the rate of heating might be important in accounting for the discrepancy. Soberman (1959) measured the critical Rayleigh number over a large range of heating rates and found that the rate of heating did indeed influence marginal stability and could increase the critical Rayleigh number considerably.

Studying the evaporative cooling of water, Spangenberg & Rowland (1961) found that the temperature profile at marginal stability was very different from that existing in the resulting flow field and that the critical temperature difference was much greater than that predicted by the existing theory. The motion which developed from the instability was found to be in the form of two-dimensional plunging sheets of cold liquid.

The foregoing experiments have established that thermally induced instabilities may occur in fluids over a range of Rayleigh numbers varying from slightly less than the theoretical value to several times the theoretical value. The actual value depends on the rate of heating, and the resulting fluid motion is apparently different in structure from the usual two-dimensional rolls.

In the present study it is attempted to establish analytically the effects of a substantially non-uniform temperature gradient on the criterion for marginal stability. The analysis is presented in §2 and the general results of the analysis are given in §3. In §4 the general results are applied to particular cases which correspond closely to the physical situations existing in the experiments mentioned above. This permits direct comparison of the present theory with the published experimental results.

2. Analysis

Consider a fluid layer which is originally isothermal and at rest. The fluid is considered infinite in horizontal extent, but finite in the vertical direction. At some time $t = 0$ heat is applied in an arbitrary manner with respect to time, but uniformly with respect to the lower bounding surface. The resulting conduction temperature profile will therefore depend on the time and the vertical spatial coordinate. The problem is to find the time after the onset of heating at which the fluid becomes unstable.

The linearized stability equation for such a situation has been derived by Goldstein (1959). In deriving the stability equation it is assumed that the arbitrary disturbance which leads to the instability may be Fourier represented. In particular, the vertical velocity component w of the disturbance and the temperature perturbation T are expanded as follows

$$w(x,y,z,t) = W(z,t)\,e^{i(k_x x + k_y y)}, \tag{1}$$

$$T(x,y,z,t) = \theta(z,t)\,e^{i(k_x x + k_y y)}, \tag{2}$$

where $k = (k_x^2 + k_y^2)^{\frac{1}{2}}$ defines the wavelength of the arbitrary disturbance. The plane $z = 0$ represents the lower extremity of the fluid, $z = h$ the upper extremity, and the co-ordinate z points in the upward vertical direction. The co-ordinates x and y lie in the horizontal plane. The dimensional equation for W may be written in the form

$$\left[\kappa\left(\frac{\partial^2}{\partial z^2}-k^2\right)-\frac{\partial}{\partial t}\right]\left[\nu\left(\frac{\partial^2}{\partial z^2}-k^2\right)-\frac{\partial}{\partial t}\right]\left(\frac{\partial^2}{\partial z^2}-k^2\right)W = \alpha g k^2 \frac{\partial T^{(0)}}{\partial z} W. \qquad (3)$$

Here, κ is the thermal diffusivity, ν is the kinematic viscosity, α is the coefficient of thermal expansion, g is the acceleration due to gravity, and $T^{(0)}$ is the unperturbed temperature which will be a function of the time t as well as the co-ordinate z. The velocity and the temperature perturbation are related by the equation

$$\left[\nu\left(\frac{\partial^2}{\partial z^2}-k^2\right)-\frac{\partial}{\partial t}\right]\left(\frac{\partial^2}{\partial z^2}-k^2\right)W = \alpha g k^2 \theta. \qquad (4)$$

The boundary conditions at the lower surface, $z = 0$, and the upper surface, $z = h$, are that the velocity and temperature perturbations should vanish. That is, on $z = 0, h$,

$$W = \frac{\partial W}{\partial z} = \left[\nu\left(\frac{\partial^2}{\partial z^2}-k^2\right)-\frac{\partial}{\partial t}\right]\left(\frac{\partial^2}{\partial z^2}-k^2\right)W = 0. \qquad (5)$$

The solution to equation (3), under the conditions (5), is complicated by the fact that $\partial T^{(0)}/\partial z$ depends on both z and t. Thus the coefficients in (3) are not constant and the variables do not separate. However, in order to determine the onset of instability it is sufficient to study the instantaneous conduction temperature profile and to ask if the fastest growing wave component of an arbitrary disturbance is growing, decaying, or is neutrally stable. Then at any instant in time, the conduction temperature profile $T^{(0)}$ will be a function of z only so that the term $\partial T^{(0)}/\partial z$ in (3) may be replaced by $dT^{(0)}/dz$ giving

$$\left[\kappa\left(\frac{\partial^2}{\partial z^2}-k^2\right)-\frac{\partial}{\partial t}\right]\left[\nu\left(\frac{\partial^2}{\partial z^2}-k^2\right)-\frac{\partial}{\partial t}\right]\left(\frac{\partial^2}{\partial z^2}-k^2\right)W = \alpha g k^2 \frac{2dT^{(0)}}{dz} W. \qquad (6)$$

It was shown by Lick (1965) that solutions to (6) of the form

$$W(z,t) = \omega^*(z)e^{\sigma t}$$

exist, where σ is real, and that these solutions correspond to physical observations. Since an arbitrary disturbance will be damped out if its fastest growing wave component is decaying, the onset of instability is defined here as the point at which the fastest growing wave component is neutrally stable. Then since σ is real, $\partial W/\partial t$ will be zero for this wave so that (6) and (5) become, respectively,

$$\left(\frac{d^2}{dz^2}-k^2\right)^3 \omega^* = \frac{\alpha g k^2}{\kappa\nu}\frac{dT^{(0)}}{dz}\omega^*; \qquad (7)$$

on $z = 0, h$,

$$\omega^* = \frac{d\omega^*}{dz} = \left(\frac{d^2}{dz^2}-k^2\right)^2 \omega^* = 0. \qquad (8)$$

The fact that $dT^{(0)}/dz$ is a function of z complicates the solution to (7). Although a series solution could be obtained, such a solution would require elaborate numerical exploration of the special functions so defined and these functions would be different for each form of $dT^{(0)}/dz$. A useful approximation, which eliminates this difficulty while still retaining the essential feature of a non-uniform temperature gradient, is to represent the actual temperature profile by

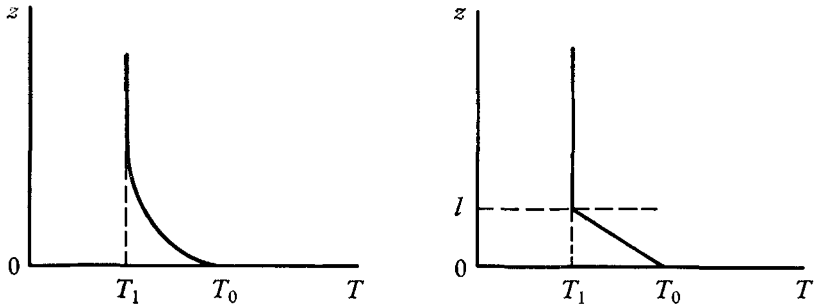

FIGURE 1. Actual temperature profile and two-segment approximation.

two straight line segments as shown in figure 1. Using the same temperature limits, the plane $z = l$ is constructed so that the area under the actual and approximate curves is the same. This same approximation was used by Lick (1965) to calculate the growth rate of disturbances in a finite fluid layer having stress-free surfaces. The depth l will be referred to as the thermal depth. Then introducing dimensionless variables $\zeta = z/h$, $\omega = (h/\kappa)\omega^*$ and $a = kh$, equation (7) gives

$$(D^2-a^2)^3\omega_1 = -a^2R\omega_1 \quad (0 < \zeta < \epsilon), \tag{9}$$

$$(D^2-a^2)^3\omega_2 = 0 \quad (\epsilon < \zeta < 1). \tag{10}$$

Here $D = d/d\zeta$, $R = \alpha g(T_0-T_1)h^4/\kappa\nu l$ is a form of Rayleigh number, T_1 is the original isothermal temperature and T_0 is the instantaneous temperature at $z = 0$, and $\epsilon = l/h$.

The boundary conditions at the interface of the two regions $\zeta = \epsilon$ are that the velocity, stresses, temperature, and heat flux are all continuous. These conditions, together with (8), result in the following conditions on ω_1 and ω_2:

$$\text{on} \quad \zeta = 0, \quad \omega_1 = D\omega_1 = (D^2-a^2)^2\omega_1 = 0; \tag{11}$$

$$\text{on} \quad \zeta = \epsilon, \quad (\omega_1-\omega_2) = (D^n\omega_1 - D^n\omega_2) = 0 \quad (n = 1,2,3,4,5); \tag{12}$$

$$\text{on} \quad \zeta = 1, \quad \omega_2 = D\omega_2 = (D^2-a^2)^2\omega_2 = 0. \tag{13}$$

The general solutions to (9) and (10) are

$$\omega_1 = C_1e^{-\gamma_1\zeta} + C_2e^{-\gamma_2\zeta} + C_3e^{-\gamma_3\zeta} + C_4e^{\gamma_1\zeta} + C_5e^{\gamma_2\zeta} + C_6e^{\gamma_3\zeta}, \tag{14}$$

$$\omega_2 = (C_7 + C_8\zeta + C_9\zeta^2)e^{-a\zeta} + (C_{10} + C_{11}\zeta + C_{12}\zeta^2)e^{a\zeta}, \tag{15}$$

where γ_1, γ_2 and γ_3 are defined by

$$\gamma_1 = a[1-(R/a^4)^{\frac{1}{3}}]^{\frac{1}{2}}, \tag{16}$$

$$\gamma_2 = a[1+\tfrac{1}{2}(1+i\surd 3)(R/a^4)^{\frac{1}{3}}]^{\frac{1}{2}}, \tag{17}$$

$$\gamma_3 = a[1+\tfrac{1}{2}(1-i\surd 3)(R/a^4)^{\frac{1}{3}}]^{\frac{1}{2}}. \tag{18}$$

Applying the boundary conditions (11), (12), (13) to the solution (14), (15) produces twelve homogeneous algebraic equations for the twelve arbitrary constants and the determinant of the coefficients of these constants is set equal to zero. That is, the following determinant is zero:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -\gamma_j & 0 & 0 & 0 \\ (\gamma_j^2-a^2)^2 & 0 & 0 & 0 \\ e^{-\gamma_j} & -e^{-a\epsilon} & -\epsilon e^{-a\epsilon} & -\epsilon^2 e^{-a\epsilon} \\ -\gamma_j e^{-\gamma_j} & a e^{-a\epsilon} & -(1-a\epsilon)e^{-a\epsilon} & -\epsilon(2-a\epsilon)e^{-a\epsilon} \\ \gamma_j^2 e^{-\gamma_j} & -a^2 e^{-a\epsilon} & a(2-a\epsilon)e^{-a\epsilon} & -(2-4a\epsilon+a^2\epsilon^2)e^{-a\epsilon} \\ -\gamma_j^3 e^{-\gamma_j} & a^3 e^{-a\epsilon} & -a^2(3-a\epsilon)e^{-a\epsilon} & a(6-6a\epsilon+a^2\epsilon^2)e^{-a\epsilon} \\ \gamma_j^4 e^{-\gamma_j} & -a^4 e^{-a\epsilon} & a^3(4-a\epsilon)e^{-a\epsilon} & -a^2(12-8a\epsilon+a^2\epsilon^2)e^{-a\epsilon} \\ -\gamma_j^5 e^{-\gamma_j} & a^5 e^{-a\epsilon} & -a^4(5-a\epsilon)e^{-a\epsilon} & a^3(20-10a\epsilon+a^2\epsilon^2)e^{-a\epsilon} \\ 0 & e^{-a} & e^{-a} & e^{-a} \\ 0 & -a e^{-a} & (1-a)e^{-a} & (2-a)e^{-a} \\ 0 & -a^4 e^{-a} & -a^4 e^{-a} & (8-a^2)e^{-a} \end{vmatrix}. \quad (19)$$

The first column represents 3 columns corresponding to $j = 1, 2, 3$. The remaining 6 columns in (19) are obtained by replacing γ_j by $-\gamma_j$ and a by $-a$.

The eigenvalues contained in (19) are the Rayleigh numbers R, and they are obtained as follows. For a given wave-number a and thermal depth ϵ, the Rayleigh number R is varied until the determinant (19) is zero. Since the elements of (19) are complex while the parameter R is real, the possibility of simultaneous vanishing of both the real and imaginary parts of (19) should be demonstrated.

Using the fact that γ_2 and γ_3 are complex conjugates and writing $\gamma_2 = \gamma_{2R} + i\gamma_{2I}$ and $\gamma_3 = \gamma_{2R} - i\gamma_{2I}$ where γ_{2R} and γ_{2I} are real, the solution (14) may be written in the form

$$\omega_1 = C_1 e^{-\gamma_1\zeta} + C_2^* e^{-\gamma_{2R}\zeta}\cos\gamma_{2I}\zeta + C_3^* e^{-\gamma_{2R}\zeta}\sin\gamma_{2I}\zeta + C_4 e^{\gamma_1\zeta} + C_5^* e^{\gamma_{2R}\zeta}\cos\gamma_{2I}\zeta + C_6^* e^{\gamma_{2R}\zeta}\sin\gamma_{2I}\zeta.$$

Then since ω_2 contains only real functions and since the boundary conditions (11), (12) and (13) involve only real functions of ω_1 and ω_2, a determinant similar to (19) could be constructed which has only real elements. That is, the roots of the complex determinant (19) may be found by varying the real parameter R.

For given a and ϵ, there is a discrete set of values of R which reduce the determinant (19) to zero. Since the smallest temperature difference which will permit a motion of the fluid is being sought, the smallest positive value of R is the required root. In this way a value of R is obtained for each-wave number a and thermal depth ϵ. The minimum value of R with respect to a is the critical Rayleigh number for that particular thermal depth ϵ.

3. General results

The iterative procedure described in §2 for reducing the determinant (19) to zero requires repeated evaluation of a twelfth-order determinant with complex elements. This was readily achieved by using a high-speed digital computer which was programmed to locate the desired roots automatically.

The variation of the critical Rayleigh number with the thermal depth ϵ is shown in figure 2, and the corresponding wave-number variation is shown in figure 3. The critical Rayleigh number R_c is 1707·8 at $\epsilon = 1$, which corresponds to the case of a uniform temperature gradient. The value of R_c has a minimum of 1340, which occurs at $\epsilon = 0{\cdot}72$, and reaches the value of 1707·8 again at $\epsilon = 0{\cdot}47$.

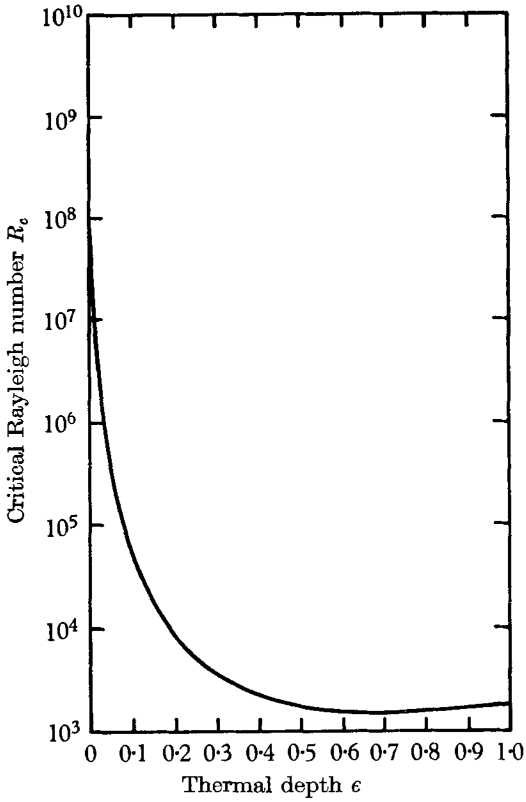

FIGURE 2. Critical Rayleigh number, $R_c = \alpha g(T_0 - T_1) h^3/\kappa\nu$, as a function of the thermal depth, $\epsilon = l/h$.

As $\epsilon \to 0$ the critical Rayleigh number, when based on the overall fluid depth h, increases without limit. However, investigation of the limit shows that if the Rayleigh number is based on the thermal depth l, the critical value is 32 as $\epsilon \to 0$. The wave-number corresponding to criticality shows no minimum, but increases monotonically from a value of 3·117 at $\epsilon = 1$ to infinity at $\epsilon = 0$.

In order to interpret these results, suppose that a fluid, which is initially isothermal and at rest, is heated from below in some manner beginning at time $t = 0$. Then initially, the Rayleigh number R and the thermal depth ϵ will both be zero. As time increases, both the Rayleigh number and the thermal depth will increase due to heat conduction in the fluid. That is, as time progresses the solution to the appropriate heat conduction problem will yield a monotonically increasing curve on figure 2. If the rate of heating is sufficiently high, the actual Rayleigh number curve will intersect the curve for the critical Rayleigh number, and beyond this point an instability will manifest itself. The Rayleigh number existing at this point is the critical Rayleigh number for that type and rate of heating. The corresponding value of ϵ permits the wave-number of the initial disturbance to be calculated from figure 3.

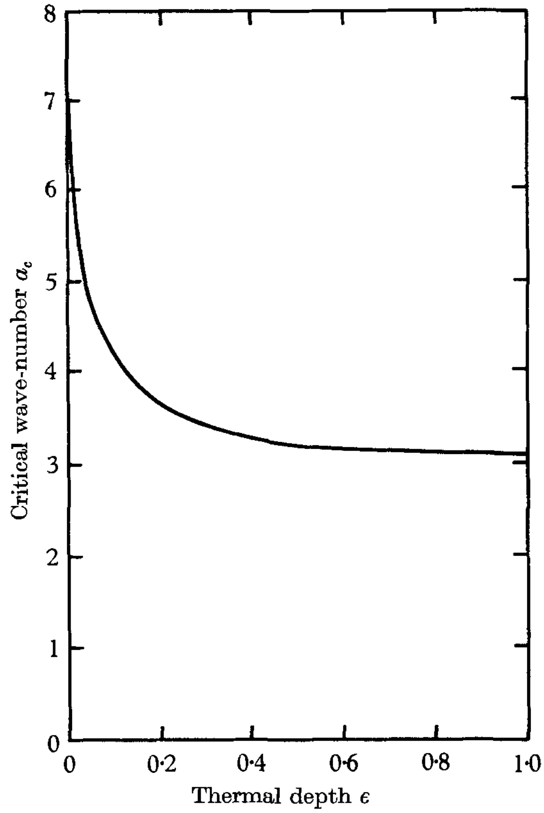

FIGURE 3. Critical wave-number, $a_c = k_c h$, as a function of the thermal depth, $\epsilon = l/h$.

There will clearly be a minimum heating rate which will just allow the heat conduction curve to intersect the critical Rayleigh number curve at $\epsilon = 1$. The time required for conduction to establish the required Rayleigh number of 1707·8 in this case will be large, and the temperature gradient at criticality will be approximately uniform. Increasing the heating rate will steepen the actual Rayleigh number curve when plotted on figure 2 so that, for a continuous variation of the heating rate, it will be possible to obtain critical Rayleigh numbers which vary from 1340 to very large values. The latter correspond to rapid rates of heating and highly non-uniform temperature gradients at criticality.

4. Comparison with experiments

It was found in §3 that the minimum value of the critical Rayleigh number is 1340. This is in close agreement with the observations of de Graaf & van der Held (1953) who found that columnar instability could be induced at Rayleigh numbers in excess of 1400.

The experimental arrangement used by Soberman (1959) was such that it corresponds closely to the idealized case of constant heating through a finite fluid layer. Then the temperature $T^{(0)}$ will be given by the solution to the heat conduction equation under the conditions that for $t < 0$ the temperature at $\zeta = 0, 1$ is T_1, while for $t > 0$ the heat flux through the plane $\zeta = 0$ is equal to a constant Q. The required solution is then,

$$\frac{q}{Qh}T^{(0)}(\zeta,\tau) = \frac{q}{Qh}T_1 + (1-\zeta) - \sum_{n=0}^{\infty}\frac{8}{(2n+1)^2\pi^2}\exp\left\{-(2n-1)^2\frac{\pi^2}{4}\tau\right\}\cos(2n+1)\tfrac{1}{2}\pi\zeta, \tag{20}$$

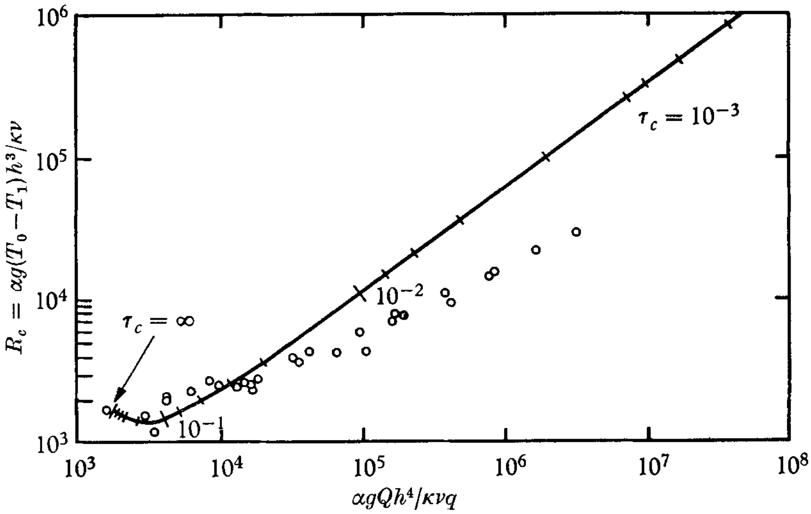

FIGURE 4. Stability curve for constant flux heating. The experimental points are due to Soberman and the dimensionless time from the onset of heating to marginal stability is $\tau_c = (\kappa/h^2)t_c$.

where q is the thermal conductivity, $\zeta = z/h$, and $\tau = (\kappa/h^2)t$. From the definitions of the Rayleigh number R and the thermal depth ϵ, equation (20) produces the following relations:

$$R(\tau) = \frac{\alpha g(T_0-T_1)h^3}{\kappa\nu} = \frac{\alpha gQh^4}{\kappa\nu q}\left\{1-\sum_{n=0}^{\infty}\frac{8}{(2n+1)^2\pi^2}\exp\left[-(2n+1)^2\frac{\pi^2}{4}\tau\right]\right\}, \tag{21}$$

$$\epsilon(\tau) = \frac{l}{h} = \frac{1-\sum_{n=0}^{\infty}\frac{(-1)^n 32}{(2n+1)^3\pi^3}\exp\left\{-(2n+1)^2\frac{\pi^2}{4}\tau\right\}}{1-\sum_{n=0}^{\infty}\frac{8}{(2n+1)^2\pi^2}\exp\left\{-(2n+1)^2\frac{\pi^2}{4}\tau\right\}}. \tag{22}$$

Equations (21) and (22) trace a curve on figure 2 as τ is varied from zero to infinity. The point at which this curve intersects the critical Rayleigh number curve depends on the magnitude of the heating parameter $(\alpha gQh^4)/(\kappa\nu q)$ appearing in equation (21). The variation of the critical Rayleigh number with this parameter is shown in figure 4 together with Soberman's results. Both theory and experiment show that the critical Rayleigh number may be considerably increased due to rapid heating.

The discrepancy between theory and experiment may be accounted for, at least in part, by the measuring technique used to determine the temperature difference $(T_0 - T_1)$ at criticality. Thermocouples were located at points $\frac{1}{8}$ in. above and below the centre line of the fluid layer, which was either $\frac{1}{2}$ in. or 1 in. deep. The overall temperature difference was calculated by linearly extrapolating the values recorded at the two measuring points. Thus, the lack of curvature in the

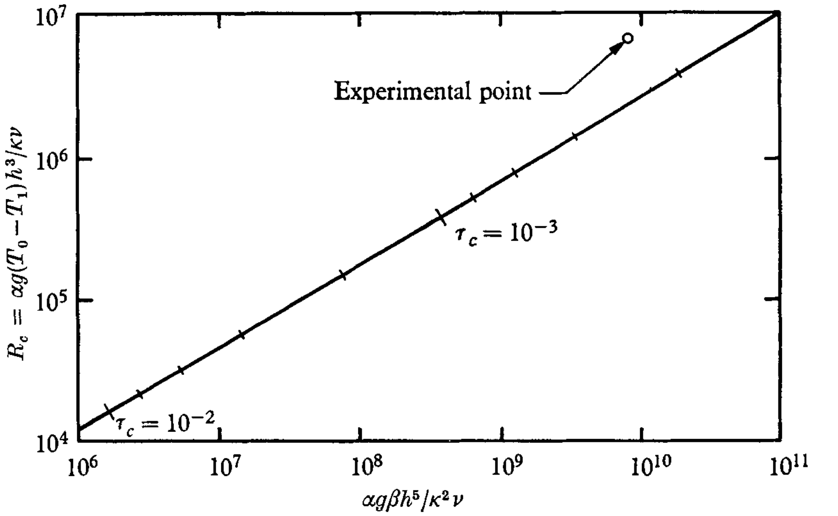

FIGURE 5. Stability curve for uniformly changing surface temperature. The experimental point is due to Spangenberg & Rowland and the dimensionless time from the onset of heating to marginal stability is $\tau_c = (\kappa/h^2)t_c$.

assumed temperature profile would result in an underestimate of the overall temperature difference and hence an underestimate of the Rayleigh number. This effect would be most pronounced at the higher heating rates.

The experiments of Spangenberg & Rowland (1961), involving sudden evaporative cooling of a layer of water, showed that the surface temperature decreased almost linearly in time up to the onset of instability. Then the temperature $T^{(0)}$ will be given by the solution of the heat conduction equation where for $t < 0$ the temperature at $\zeta = 0, 1$ is T_1, while for $t > 0$ the temperature at $\zeta = 1$ is decreasing by an amount βt. Solving the heat conduction equation and using the definitions of R and ϵ results in the following expressions:

$$R(\tau) = \frac{\alpha g(T_0 - T_1)h^3}{\kappa\nu} = \frac{\alpha g\beta h^5}{\kappa^2\nu}\tau, \tag{23}$$

$$\epsilon(\tau) = \frac{l}{h} = 1 - \frac{8}{\tau}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4\pi^4}[1 - e^{-(2n+1)^2\pi^2\tau}], \tag{24}$$

where $\tau = (\kappa/h^2)t$. The point of intersection of the curve defined by (23) and (24) with the curve of figure 2 depends on the magnitude of the parameter $(\alpha g\beta h^5)/(\kappa^2\nu)$ which appears in (23). The variation of the critical Rayleigh number with this parameter is shown in figure 5. Since the rate of evaporation and hence the rate of cooling was not controlled in Spangenberg & Rowland's experiments, only one experimental point was obtained. This point is also shown in figure 5. Both theory and experiment agree in the fact that the critical Rayleigh number is increased by almost four orders of magnitude. The theoretical time for the onset of instability is 20 s after the onset of cooling, whereas the observed time was 70 s.

The fact that the experimental point lies above the theoretical curve in figure 5 may be attributed, in part, to the definition of the critical Rayleigh number. The onset of instability was defined by Spangenberg & Rowland as the point at which a motion of the fluid was first observed. In the analysis, the onset of instability was defined as the point at which the fastest growing wave becomes marginally stable. Thus a certain time must elapse before the disturbance develops from an infinitesimal amplitude to a perceptible fluid motion. During this time the surface temperature of the fluid continues to decrease uniformly so that the Rayleigh number continues to increase. Thus, the fact that the experimental values of R_c and τ_c are greater than the theoretical values is consistent and in qualitative agreement with the circumstances.

The experiments discussed above are the only experiments known to the author in which numerical data for the onset of instability are recorded for an anomalous mode of motion. However, data exist for the minimum Rayleigh number of the sustained form of columnar motion as recorded by Chandra (1938) and by Sutton (1950). These Rayleigh numbers are smaller than the values predicted by the classical theory.

In order to explain one of the fundamental differences between rolling and columnar instability, consider first a fluid which is heated slowly so that the temperature gradient is approximately uniform. Then from figure 3 it is seen that the wave-number of the initial disturbance will be 3·117, since ϵ will be close to unity. Thus, as time progresses beyond τ_c, the value corresponding to marginal stability, the value of ϵ, and hence the wave-number will remain unchanged. That is, the same wave component will be the fastest growing throughout the growth time so that the initially unstable wave will have approximately the same form as the final flow field, although the latter may be distorted due to finite amplitude effects.

Now consider rapid heating. There will be some wave-number a_c of the initially unstable wave, with $a_c > 3{\cdot}117$. The corresponding value of ϵ will be less than unity. Now as time progresses beyond τ_c, conduction will continue so that ϵ will continue to increase. Thus the wave-number of the fastest growing disturbance will be continuously changing. This means that the form of the final flow field will probably bear very little resemblance to the form of the initial instability.

From experiments it is known that in columnar motion the rising fluid occupies a much smaller flow area than the descending fluid, or vice versa. Furthermore, the temperature profile is distorted substantially from the approximately linear variation which exists in the two-dimensional rolls. These facts would seem to indicate that the columnar mode is a non-linear mode and so the linear methods which have been so successful in analysing the rolling motion would not be expected to succeed in analysing columnar motion.

5. Concluding remarks

The effect of a strongly non-uniform temperature gradient on the criterion for marginal stability has been established. It has been shown analytically that the method and rate of heating may greatly influence the value of the Rayleigh number existing at marginal stability. For slow heating, the critical Rayleigh number for a fluid with two rigid boundaries agrees with the classical value of 1707·8. However, at higher heating rates, the critical Rayleigh number may be as low as 1340 or it may be increased indefinitely. The wave-number of the initially unstable wave will always be greater than the classical value of 3·117 when the heating is not slow. The findings on marginal stability are in good qualitative agreement with the published experimental findings.

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